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The irreducible soluble subgroup of GL(2,p^k)

Behnam Razzaghmaneshi

Assistant professor of Department of Mathematics Talesh Branch, Islamic Azad University, Talesh, Iran

Corresponding author: Behnam Razzaghmaneshi

ABSTRACT: this paper we complete and irredundant list of $GL(2,p^k)$-conjugacy class representatives for $p^k = 1,...30$ of the primitive soluble subgroups of $GL(2,p^k)$, for $p^k = 1,...30$, whose guardian is M_3 or M_4 of $GL(2,p^k)$.

Keywords: JS-maximale, Soluable, Transitive.

INTRODUCTION

Let F be the field of p^k elements.Since $GL(1,F)$ and S_2 are soluble, there is exactly one **JS**–imprimitive of $GL(2,F)$, namely $M_1 := GL(1,F)\ wr\ S_2, (p^k \neq 2)$

Theorem C provides a complete and irredundant list l_1' of $GL(2,F)$-conjagacy class representatives of the non–cyclic irreducible subgroups of M_1.

Now consider the **JS**–primitives of $GL(2,F)$.The unique maximal abelian normal subgroup of such a group has order either $p^{2k}-1$ or p^k-1.There is just one **JS**-primitive of the first kind, namely

$$M_2 := C_{p^{2k}-1} \succ C_2,$$

The normaliser of a singer cycle. Theorem E provides a complete and irredundant list l_2' of $GL(2,F)$-conjugacy class representatives of the primitive subgroups and imprimitive cyclic subgroups of M_2.

There are **JS**–primitives whose unique maximal abelian normal subgroup has order p^k-1 if and only if p is odd.By remark 2.36, they can be written as

$$M_3 := (C_{p^k-1} Y\ \ Q_8)\ \ N\ \ O^-(2,2),\ p^k \equiv 3(\mathrm{mod}4),$$

$$M_4 := (C_{p^k-1} Y\ \ Q_8)\ \ N\ \ Sp(2,2),\ p^k \equiv 1\ (\mathrm{mod}4).$$

In this chapter we give theorems which provide complete and irredundant lists $l_i', i=3,4$ of $GL(2,F)$–conjugacy class represcentatives of the primitive soluble subgroups of $GL(2,F)$ whose guardian is M_i.

Then the combination of l_1', l_2' and l_i' gives us a complete and irredundantlist of $GL(2,F)$-cojugacy class representatives of the irreducible soluble subgroups of $GL(2,F)$.

We mention in passing which of the above **JS**–maximals are maximal soluble subgroups of $GL(2,F)$. Both jordan (1871a, table A, p.288) and suprunenko (1976, footnote on p.165) knew the following result; It is easily proved using elementary number theory and some basic structural properties of the group concerned.

1.1-Proposition :

(a) M_1 is a maximal soluble subgroup of $GL(2,F)$ except when $p^k=3$ (in which case M_1 is conjugate to subgroups of both M_2 and M_3) and when $p^k=5$ (in which case M_1 is conjugate to a subgroup of M_4).

(b) M_2 is a maximal soluble subgroup of $GL(2,F)$ except when $p^k=3$ (in which case M_2 is subgroup of M_3).

(c) M_3 is a maximal soluble subgroup of $GL(2,F)$.

(d) M_4 is a maximal soluble subgroup of $GL(2,F)$.

1.2-Proposition :

Let V be a vector space, and A be the scaler subgroup of $GL(V)$. Let G be a subgroup of $GL(V)$ cantaining A, and suppose that $G=NA$ for some subgroup N of G. Let H be a subgroup of G cantaining $N\cap A$. Then,

(a) H is irreducible if and only if $N\cap HA$ is irreducible;

(b) H is primitive if and only if $N\cap HA$ is primitive.

Proof :

By Dedkind's modular law we have that $HA=(N\cap HA)A$.
Since multiplication by scalars dose not effect the reducibility or primitivity of a group, the result now follows.

1.3-Definition:

The Burnside poset $P,$ of a group G is a poset with the following properties.

(i) Each element of P represents a conjugacy class of subgroups of G and each conjugacy class is represented exactly once in P .

(ii) If C and D are elements of P, then we write $C\leq D$ if and only if at least one group in the conjugacy class represented by C is a subgroup of at least one group in the conjugacy class represented by D.

1.4-Definition:

We draw the Burnside inclusion diagram of the Burnside poset P of a group G according to the following rules .

(i) We represent elements of P by small black discs.

(ii) If an element of P represents a conjugacy class cantaining a single group (which is therefore normal in G), we draw a circle around the disc representing that element.

(iii) We lable each disc with either the isomorphism type or the order of the groups in the conjugacy class it represents.

(iv) We represent the relation $C\leq D$ by placing the disc representing C lower on the page then the disc reperesenting D and by drowing a line between those two discs. Howerer, we suppress inclusion that are implied by the reflexiveness and transitivity of $\leq$.

2-Generating sets for M_3 and M_4.

In this section we derive polycyclic presentations for M_3 and M_4.

2.1-A generating set for M_3.

Recall that M_3 only defind when $p^k \equiv 3(\mathrm{mod}4)$ .We construct a generating set for M_3 (by the methods and theorems described in chapter 2).Let z be a generator for the scalar group, and defind u and v by $u := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $v := \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$

where α and β belong to the prime subfield of F, and $\alpha^2 + \beta^2 = -1$.

Then $\{u, v, z\}$ generates the Fitting subgroup of M_3.To extend this set to a generating set for M_3, we first require a generating set for $O^-(2,2)$.The set we use comprises the two elements $a\rho := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $c\rho := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, these matrices satisfy the conditions, $(a\rho)^2 = I_2$,$(a\rho c\rho)^3 = I_2$,$(a\rho c\rho)^{a\rho} = (a\rho c\rho)^2$. By the theory in chapter **2** there exist matrices a and c of $GL(2,F)$ satisfying the conditions

$u^a = \lambda_1 v$,

$v^a = \mu_1 u$,

$u^c = \lambda_2 uv,$

$v^c = \mu_2 v,$ for some scalars λ_1, λ_2 , μ_1 and μ_2.

Let δ be an element of F such that $\delta^2 = \begin{cases} -2 & \text{if } p^k \equiv 3(\mathrm{mod}8), \\ 2 & \text{if } p^k \equiv 7(\mathrm{mod}8). \end{cases}$

Setting $\lambda_1 = \mu_1 = -1$, we find that one solution for a is

$$a := \delta^{-1} \begin{bmatrix} \alpha & \beta + 1 \\ \beta - 1 & -\alpha \end{bmatrix}.$$

Then a has determinat -1 or 1, and its square is I_2 or $-I_2$, according as p^k is congruent to 3 or 7 modulo 8, repectively.Setting $\lambda_2 = \mu_2 = -1$, we find that one solution for c is $c := \pm\delta^{-1} \begin{bmatrix} -\beta & \alpha + 1 \\ \alpha - 1 & \beta \end{bmatrix}$,

the minus or plus being chosen according as p^k is congrunt to **3** or **7** modulo **8**, respectively, then c has determinant -1 or 1, and its square is I_2 or -I_2, according as p^k is congrunt to **3** or **7** modulo **8**, respectively.Set $b := ac$.Then

$$b = 2^{-1} \begin{bmatrix} \alpha - \beta - 1 & \alpha + \beta - 1 \\ \alpha + \beta - 1 & -\alpha - \beta - 1 \end{bmatrix}$$

And b has determinal **1** and order **3**. Furthermore, $b^a = b^2$ .We then have the following presentation for M_3:

$$\{a, b, u, v, z \mid a^2 = \pm I_2 ,$$

$$b^a = b^2 \quad , \; b^3 = I_2 ,$$
$$u^a = -v \quad , u^b = v \quad , u^2 = -I_2 \; ,$$
$$v^a = -u \quad , \; v^b = uv , v^u = -v \quad , v^2 = -I_2 ,$$
$$z^a = z \quad , z^b = z \;\; , z^u = z \quad , z^v = z \quad , z^{p^k-1} = I_2 \; \}.$$

The plus or minus being present in the first relation according as p^k is congruent to **3** or **7** modulo **8**' respectively.This is clearly a polycyclic presentation after reaplacing -I_2 by $z^{(p^k-1)/2}$.From this presentation we see that

$$M_3 = <z> Y < a,b,u,v>$$
$$= <z^2> \times <a,b,u,v>$$
$$\cong \begin{cases} C_{(p^k-1)/2} \times GL(2,3) & \text{if } p^k \equiv 3 (\text{mod} 8), \\ C_{(p^k-1)/2} \times BO & \text{if } p^k \equiv 7 (\text{mod} 8) \end{cases}$$

Note that Wilson (1972' theorem 3.2' p.36) also observed that M_3 splits over its Fitting subgroup when F has a square root of -2 (that is' when $p^k \equiv 3 (\text{mod} 8)$).

It is easly to show that $<a,b,u,v>$ is the unique minimal supplement to the scalar group' and that

$$M \cap SL(2,F) = \begin{cases} <b,u,v> & \text{if } p^k \equiv 3 (\text{mod} 8), \\ <a,b,u,v> & \text{if } p^k \equiv 7 (\text{mod} 8) \end{cases}$$
$$\cong \begin{cases} SL(2,3) & \text{if } p^k \equiv 3 (\text{mod} 8), \\ BO & \text{if } p \equiv 7 (\text{mod} 8). \end{cases}$$

Finally' we inrestigate action of field a automorphisms on M_3.

2.2-Theorem:

Every automorphism of F, acting entry-wise on the elements of M_3, normaliseres M_3.

Proof:

Let θ be an automorphism of F.By looking at the entrise of the matrices in our generating set for M_3, it is clear that the effect of θ on M_3 is determined by $\delta\theta$ (remember that α and β belong to the prime subfield of F) cleary b,u and v are fixed by θ and $a\theta$ is a or $-a$.And ofcourse $z\theta$ is some power of itself. Therefore θ normalises M_3.

3-A generating set for M_4 :

Recall that M_4 is only defined when $p^k \equiv 1 (\text{mod} 4)$. We construct a generating set for M_4 in this section by use of preceding methods. Let z be a generator for the scalar group' and define u and v by

$u := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $v := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

then $\{u, v, z\}$ generates the Fitting subgroup of M_4.To extend this set to a generating set for M_4.We first require a generating set for $Sp(2,2)$.We choose this set to consist of the two elements

$a\rho := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $b\rho := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

these matrices satisfy the relations

$$(a\rho)^2 = I_2, (b\rho)^3 = I_2, (b\rho)^{a\rho} = (b\rho)^2.$$

By use from chapter 2 there exist matrices a and b of $GL(2,F)$ satisfying the conditions

$$u^a = \lambda_1 v \ ,$$
$$v^a = \mu_1 u \ ,$$
$$u^b = \lambda_2 v \ ,$$
$$v^b = \mu_2 uv.$$

For some scalars $\lambda_1, \lambda_2, \mu_1$ and μ_2.

Let w be a primitve **4**-th-root of unity in F ' and let δ be an element of F such that

$$\delta^2 := \begin{cases} -2 \ \ if \ \ p^k \equiv 1(\mathrm{mod}8), \\ \ \ 2 \ \ if \ \ p^k \equiv 5(\mathrm{mod}8). \end{cases}$$

Setting $\lambda_1 = \mu_1 = -1$, we find that one solution for a is

$$a := \delta^{-1}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

If $p^k \equiv 1(\mathrm{mod}8),$ then a has determinat 1 and its square is -I_2.

If $p^k \equiv 5(\mathrm{mod}8),$ then a has determinat -ω and its square is ωI_2.

Setting $\lambda_2 = 1$ and $\mu_2 = \omega,$

We find that on solution for b is $b := 2^{-1}\begin{bmatrix} \omega - 1 & \omega + 1 \\ \omega - 1 & -\omega - 1 \end{bmatrix}$

Then b has determinat 1 and order 3' and $b^a = b^2.$

We then the following presentation for M_4:

$$\{a, b, u, v, z \,|\, a^2 = \sigma I_2,$$

$$b^a = b^2, b^3 = I_2,$$
$$u^a = -v, u^b = v, u^2 = I_2,$$
$$v = -u, v^b = \omega uv, v^u = -v, v^2 = I_2,$$
$$z^a = z, z^b = z, z^u = z, z^v = z, z^{p^k-1} = I_2\},$$

where σ is –1 or w, according as p^k is congruent to 1 or 5 modulo 8, respectively.Since b dosenot normalise $< u, v >$, it is more convernient to work $< \omega u, \omega v >$. Setting $x := \omega u \quad y := \omega v$, then we have

$$\{a, b, x, y, z \mid a^2 = \sigma I_2,$$
$$b^a = b^2, b^3 = I_2,$$
$$x^a = -y, x^b = y, x^2 = -I_2,$$
$$y^a = -x, y^b = xy, y^x = -y, y^2 = -I_2,$$
$$z^a = z, z^b = z, z^x = z, z^y = z, z^{p^k-1} = I_2\},$$

which is clearly a polycyclic presentation for M_4(after replacing -I_2 by $z^{\frac{(p^k-1)}{2}}$ and, if necessary, ωI_2 by $z^{\frac{(p^k-1)}{4}}$). From this repersentation we see that

$$M_4 = < z > \Upsilon < a, b, x, y >$$
$$= \begin{cases} < z > \Upsilon < a, b, x, y >, & if \quad p^k \equiv 1(\bmod 8), \\ < z^4 > \times < a, b, x, y >, & if \quad p^k \equiv 5(\bmod 8). \end{cases}$$

$$\cong \begin{cases} C_{p^k-1} \Upsilon \ BO & if \quad p^k \equiv 1(\bmod 8), \\ C_{\frac{(p^k-1)}{4}} \times NS & if \quad p^k \equiv 5(\bmod 8). \end{cases}$$

It is easy to show that $< a, b, x, y >$is a minimal supplement to the scalar group. If $p^k \equiv 5(\bmod 8),$ then it is the unique such supplement.

If $p^k \equiv 1(\bmod 8),$ then there is just one other minimal supplement to the scalar group; it is< $\omega a, b, x, y$ >, which is isomorphic to $GL(2,3)$.

Note that wilson (1972, theorem 3.2, p.36) also observed that M_4

Splits over its Fitting subgroup when F has a square root of -2 (that is, when $p^k \equiv 1(\bmod 8)$), Also, we have that

$$M_4 \cap SL(2, F) = \begin{cases} < a, b, x, y > & if \ p^k \equiv 1(\bmod 8) \quad , \\ < b, x, y > & if \quad p^k \equiv 5(\bmod 8). \end{cases}$$
$$\cong \begin{cases} BO & if \ p^k \equiv 1(\bmod 8) \quad , \\ SL(2,3) & if \ p^k \equiv 5(\bmod 8) \quad . \end{cases}$$

Finally, we investigate the action of field automorphism on M_4.

***3.1-Theorem**:*

Every automorphism of F, acting entry-wise on the elements of M_4, normalises M_4.

Proof:

Let θ be an automorphism of F. By looking at the entries of the matrices in our generating set for M_4, it is clear that the effect of θ on M_4 is determined by $\omega\theta$ and $\delta\theta$. It is easly to check that, in any case, $x\theta$ is $\pm x$, $y\theta$ is $\pm y$, $a\theta$ is $\pm \omega\theta$ and $b\theta$ is b or by.(and ofcourse $z\theta$ is some power of itself). Therefore θ normalises M_4.

4.5-soluble subgroups of** $GL(2,p^k)$ **for** p^k **=1, …, 30 .

We knew that the **JS**-maximals of $GL(2,p^k)$ as follows

$$M_1(2,F) := GL(1,p^k) \ wr \ S_2, p^k \neq 2;$$
$$M_2(2,F) := C_{p^{2k}-1} \succ C_2;$$
$$M_3(2,F) := (C_{p^k-1} \text{Y} Q_8 \quad \text{N} \quad \overline{O}(2,2), p^k \equiv 3(\text{mod}4);$$
$$M_4(2,F) := (C_{p^k-1} \text{Y} Q_8) \quad \text{N} \quad Sp(2,2), p^k \equiv 1(\text{mod}4).$$

Therefore for $p^k = 1,...,30$, we will have a following table.

$GL(2,2)$		M_2		
$GL(2,3)$	M_1	M_2	M_3	
$GL(2,4)$	M_1	M_2		
$GL(2,5)$	M_1	M_2		M_4
$GL(2,6)$	M_1	M_2		
$GL(2,7)$	M_1	M_2	M_3	
$GL(2,8)$	M_1	M_2		
$GL(2,9)$	M_1	M_2		M_4
$GL(2,10)$	M_1	M_2		
$GL(2,11)$	M_1	M_2	M_3	
$GL(2,12)$	M_1	M_2		
$GL(2,13)$	M_1	M_2		M_4
$GL(2,14)$	M_1	M_2		
$GL(2,15)$	M_1	M_2	M_3	
$GL(2,16)$	M_1	M_2		
$GL(2,17)$	M_1	M_2		M_4
$GL(2,18)$	M_1	M_2		
$GL(2,19)$	M_1	M_2	M_3	
$GL(2,20)$	M_1	M_2		
$GL(2,21)$	M_1	M_2		M_4
$GL(2,22)$	M_1	M_2		

$GL(2,23)$	M_1	M_2	M_3	
$GL(2,24)$	M_1	M_2		
$GL(2,25)$	M_1	M_2		M_4
$GL(2,26)$	M_1	M_2		
$GL(2,27)$	M_1	M_2	M_3	
$GL(2,28)$	M_1	M_2		
$GL(2,29)$	M_1	M_2		M_4
$GL(2,30)$	M_1	M_2		

Now we can obtained by use Theorems A,B,C and D,a complete and irredundant list of $GL(2,p^k)$-conjugacy class representatives for $p^k = 1,\dots 30$ of the primitive soluble subgroups of $GL(2,p^k)$, for $p^k = 1,\dots 30$, whose guardian is M_3 or M_4.Or

We can obtain a form of complete and irredundant set of $GL(2,p^k)$-conjugacy class representatives of the primitive subgroups and cyclic imprimitive subgroups of M_2 and for odd p, complete and irredundant set of $GL(2,p^k)$-conjugacy class representative of the non-abelian imprimitive soluble subgroups of $GL(2,p^k)$.

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